On the relationships between theories of time granularity and the monadic second-order theory of one successor

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ABSTRACT. *In this paper we explore the connections between the monadic second-order theory of one successor (*$\mathrm{MSO}[<]$ *for short) and the theories of* ω*-layered structures for time granularity. We first prove that the decision problem for* $\mathrm{MSO}[<]$ *and that for a suitable first-order theory of the upward unbounded layered structure are inter-reducible. Then, we show that a similar result holds for suitable chain variants of the MSO theory of the totally unbounded layered structure (this allows us to solve some decision problems about theories of time granularity left open by Franceschet et al. [FRA 06]).*

KEYWORDS: *time granularity, the sequential calculus, decidability.*

1. Introduction

The ability of representing the same situation and/or different situations at various time granularities and of properly relating these different representations is a long-standing research theme for temporal logic and a major requirement for a number of applications in different areas of computer science, including formal methods, artificial intelligence, and temporal databases, e.g., [BET 00, DYR 95, FIA 94, LAD 86, LAM 85]. In particular, in the area of specification and automatic verification of complex systems [FRA 04, MON 99, MON 02, MON 96b], the addition of a notion of time granularity makes it possible to specify in a concise way reactive systems whose behaviour can be naturally modeled with respect to a (possibly infinite) set of differently-grained temporal domains.

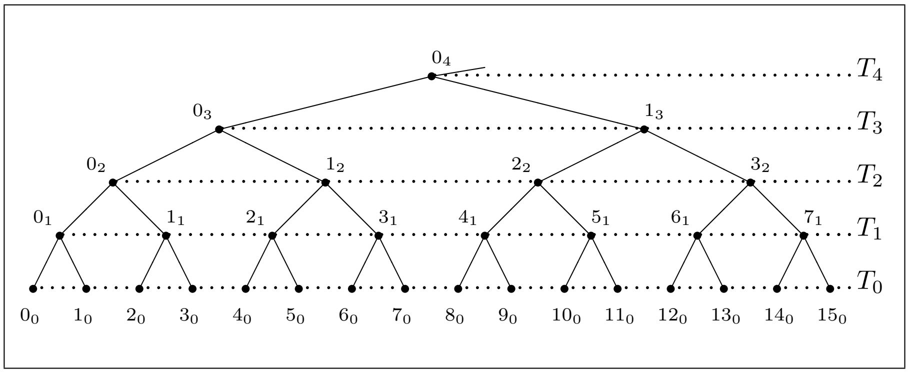

Figure 1. *The* 2*-refinable upward unbounded layered structure*

A logical framework for *time granularity* has been proposed in [MON 96a] and later extended in [FRA 01, PUP 06]. It is based on a many-level view of temporal structures that replaces the flat temporal domain of standard linear and branching temporal logics with a temporal universe consisting of a (possibly infinite) set of differently-grained temporal domains.

The monadic second-order (MSO) theory of the n-layered (there are exactly n temporal domains) k-refinable (each time point can be refined into k time points of the immediately finer temporal domain, if any) temporal structure for time granularity, with matching decidability results, has been investigated in [MON 96b]. The MSO theory of the k-refinable *upward unbounded layered structure* (UULS, for short), that is, the ω-layered structure consisting of a finest temporal domain together with an infinite number of coarser and coarser domains (a portion of the 2-refinable UULS is depicted in Figure 1) has been studied in [MON 99]. In the same paper, the authors deal with the MSO theory of the k-refinable *downward unbounded layered structure* (DULS), that is, the ω-layered structure consisting of a coarsest domain together with an infinite number of finer and finer domains (a portion of the 2-refinable DULS is depicted in Figure 2). Finally, the MSO theory of the k-refinable *totally unbounded layered structure* (TULS), which can be viewed as the merging of the UULS and the DULS (a portion of the 2-refinable TULS is depicted in Figure 3), has been studied in [PUP 06].

The decidability of the MSO theory of the UULS can be proved by reducing the satisfiability problem for MSO logic over the UULS to the emptiness problem for systolic tree automata, while the decidability of the MSO theories of the DULS and the TULS can be proved by reducing the satisfiability problem for MSO logic over them to the emptiness problem for Rabin tree automata. The structure of the decidability proofs for the UULS and the DULS is briefly summarized in [EUZ 05]. The proof for the TULS is an easy adaptation of the one for the DULS. The proof for the DULS exploits an embedding technique that appends the infinite sequence of k-refinable infinite trees of the DULS to the rightmost branch of the $(k + 1)$-ary tree (and then interprets it into the k-ary tree). The same technique can be applied to the TULS, provided that we reverse the edges on the leftmost branch from a given node upward. In [MON 04], Montanari and Puppis show that one can embed the UULS (resp., DULS) into the TULS by adding a unary predicate, called *layer 0* predicate, that identifies a distinguished layer of the structure, namely, the bottom (resp., top) layer of the UULS (resp., DULS). The decision problem for such an expanded structure has been solved by reducing it to the acceptance problem for Rabin tree automata.

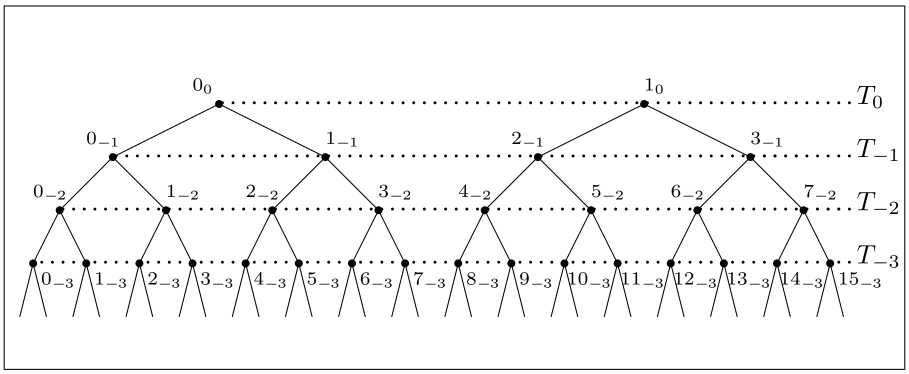

Figure 2. *The* 2*-refinable downward unbounded layered structure*

In this paper, we establish some interesting connections between the MSO theory of one successor (often called the sequential calculus), denoted by $\mathrm{MSO}[<]$, and suitable fragments of (variants of) MSO theories of ω-layered structures. In particular, we take into consideration ω-layered structures expanded with the equi-level and equi-column predicates. The equi-level predicate constrains two time points to belong to the same layer, while the equi-column predicate constrains them to be at the same distance from the origin of the layers they belong to. Definability and decidability issues for ω-layered structures expanded with the equi-level and equi-column predicates have been systematically investigated by Franceschet et al. in [FRA 06][1].

Here we broaden the scope of such an investigation. First, we introduce a notion of reducibility via interpretation and we exploit it to compare the first-order (FO) and MSO logics of the discrete linear order $\langle \mathbb{N}, < \rangle$ with those of the 2-refinable UULS (all results can be generalized to any $k > 2$). One can easily show that the FO (resp., MSO) logic of $\langle \mathbb{N}, < \rangle$ can be interpreted into the FO (resp., MSO) logic of the 2-refinable UULS, but not vice versa. We prove that the FO (resp., MSO) logic of the 2-refinable UULS can be interpreted into the FO (resp., MSO) logic of $\langle \mathbb{N}, < \rangle$ expanded with the binary predicate (actually a function) $flip$, which expresses properties of the binary representations of numbers [MON 00b]. Next, we introduce a relaxed notion of reducibility, which allows us to define a mapping of formulas (and valuations) from one logic to another one where each variable can be mapped into one or more variables

1. In [FRA 06], the authors also provide a succinct account of existing results about definability and decidability problems for k-ary trees expanded with equi-level and equi-column predicates.

of possibly different types (individual and set variables). More precisely, we define (i) a function α that maps every formula ϕ of a logic $\mathcal{L}'$ over a relational structure $\mathcal{S}'$ into a formula $\alpha(\phi)$ of a logic $\mathcal{L}$ over a relational structure $\mathcal{S}$ and (ii) a function γ that maps a valuation θ for the variables in ϕ into a valuation $\gamma(\theta)$ for the variables in $\alpha(\phi)$ such that $\mathcal{S}', \theta \models \phi$ if and only if $\mathcal{S}, \gamma(\theta) \models \alpha(\phi)$.

As a first result, we define a mapping of formulas of MSO logic over $\langle \mathbb{N}, < \rangle$ into formulas of FO logic over the 2-refinable UULS, expanded with a pair of suitable predicates $Path^<$ and D_0, and a corresponding mapping of valuations, which basically encodes finite sets into natural numbers, that make it possible to reduce the satisfiability problem for the first logic to that for the second one. We also provide the converse reduction, thus showing that the satisfiability problems for the two logics are actually inter-reducible. As a matter of fact, the latter reduction allows one to map verification problems for the FO logic of the UULS to verification problems for the MSO logic of $\langle \mathbb{N}, < \rangle$, thus making it possible to exploit the wide spectrum of verification techniques and tools available for that logic.

Then we consider the TULS equipped with the *layer* 0 predicate, denoted by T_0, and either the equi-level predicate, denoted by T, or the equi-column one, denoted by D. We exploit a different encoding of (possibly infinite) chains, that is, subsets of paths, in order to reduce the satisfiability problem for the chain fragment of MSO logic over the 2-refinable TULS to the satisfiability problem for (full) MSO logic over $\langle \mathbb{N}, < \rangle$. The converse reduction is accomplished by embedding $\langle \mathbb{N}, < \rangle$ into the leftmost branch of the TULS.

All together these results enlighten the relationships between the MSO theory of one successor and various theories of ω-layered structures. In addition, the characterization of the chain fragment of MSO logic over the TULS, expanded with T_0 and D, positively answers to some decision problems left open in [FRA 06], namely, the problem of establishing whether the satisfiability problem for the chain/path/first-order fragments of MSO logic over the 2-refinable DULS, expanded with D, is decidable.

The rest of the paper is organized as follows. In Section 2 we introduce background knowledge and notation, and we recall basic results about the MSO logics of ω-layered structures. In Section 3 we compare FO and MSO logics interpreted over $\langle \mathbb{N}, < \rangle$ and the binary UULS. In Section 4 we show that the satisfiability problems for MSO logic over $\langle \mathbb{N}, < \rangle$ and for FO logic over the 2-refinable UULS, expanded with $Path^<$ and D_0, are inter-reducible. In Section 5 we prove that the satisfiability problems for MSO logic over $\langle \mathbb{N}, < \rangle$ and for the chain fragment of MSO logic over the TULS, expanded with T_0 and either T or D, are inter-reducible. Conclusions summarize the achieved results and outline future research directions.

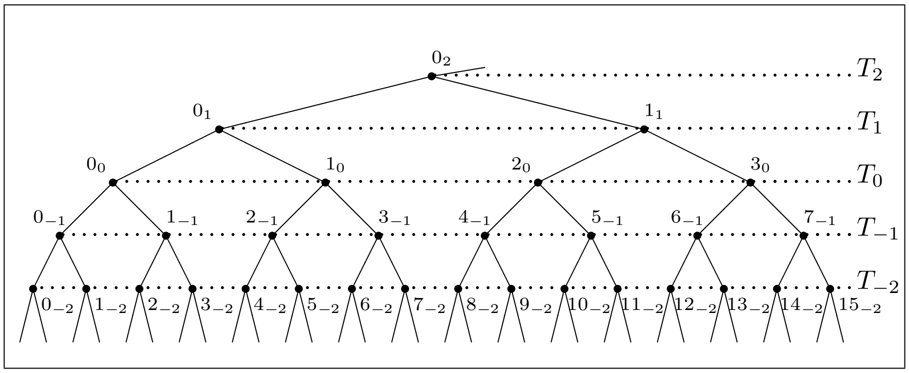

Figure 3. *The* 2*-refinable totally unbounded layered structure*

2. Logics of ω-layered structures

In this section we introduce the logics and the relational structures we are interested in. Moreover, we briefly describe the logical tools that will be used to explore their relationships.

DEFINITION 1 (THE LANGUAGE OF MSO LOGIC). — *Let $\tau = c_1, ..., c_r, u_1, ..., u_s, b_1, ..., b_t$ be a finite alphabet of relational symbols, where $c_1, ..., c_r$ (resp. $u_1, ..., u_s$, $b_1, ..., b_t$) are constant symbols (resp. unary relational symbols, binary relational symbols), and let $\mathcal{P}$ be an alphabet of (uninterpreted) unary relational symbols. The language $\mathrm{MSO}[\tau \cup \mathcal{P}]$ of MSO logic over τ and $\mathcal{P}$ is defined as follows:*

– atomic formulas *are of the forms $x = y$, $x = c_i$, with $1 \leq i \leq r$, $u_i(x)$, with $1 \leq i \leq s$, $b_i(x, y)$, with $1 \leq i \leq t$, $x \in X$, and $P(x)$, where x, y are individual variables, X is a set variable, and $P \in \mathcal{P}$;*

– formulas *are built up from atomic formulas by means of the Boolean connectives $\neg$ and $\wedge$, and the quantifier $\exists$ ranging over both individual and set variables.*

In the following, we shall write $\mathrm{MSO}_{\mathcal{P}}[\tau]$ for $\mathrm{MSO}[\tau \cup \mathcal{P}]$ and we shall write $\mathrm{MSO}[\tau]$ when $\mathcal{P}$ is meant to be the empty set.

The symbols belonging to the signature τ are interpreted over a suitable relational structure, such as, for instance, the set $\mathbb{N}$ of natural numbers or an infinite tree, in the obvious way. Details can be found in [THO 97]. The satisfiability problem for $\mathrm{MSO}[\tau]$ (resp., $\mathrm{MSO}_{\mathcal{P}}[\tau]$) with respect to a given relational structure is the problem of establishing, for any given $\mathrm{MSO}[\tau]$-formula (resp., $\mathrm{MSO}_{\mathcal{P}}[\tau]$-formula) $\phi(x_1, ..., x_m, X_1, ..., X_n)$, whether there exists a valuation of the free variables $x_1, ..., x_m, X_1, ..., X_n$ (resp., of free variables $x_1, ..., x_m, X_1, ..., X_n$ and the symbols in $\mathcal{P}$) that satisfies ϕ. The $\mathrm{MSO}[\tau]$ (resp., $\mathrm{MSO}_{\mathcal{P}}[\tau]$) theory of a given relational structure (resp., $\mathcal{P}$-labeled relational structure) is the set of all and only the $\mathrm{MSO}[\tau]$-sentences (resp., $\mathrm{MSO}_{\mathcal{P}}[\tau]$-sentences) that hold in the structure (resp., $\mathcal{P}$-labeled structure). The decision problem for the $\mathrm{MSO}[\tau]$ (resp., $\mathrm{MSO}_{\mathcal{P}}[\tau]$) theory of a given structure (resp., $\mathcal{P}$-labeled structure) can be easily reduced to the satisfiability problem for $\mathrm{MSO}[\tau]$ (resp., $\mathrm{MSO}_{\mathcal{P}}[\tau]$) over such a structure (resp., $\mathcal{P}$-labeled structure). For this reason, hereafter we shall concentrate our attention on the latter problem.

In the following, we shall also take into consideration the first-order fragment $\mathrm{FO}[\tau]$ of $\mathrm{MSO}[\tau]$, over ω-layered structures, as well as its path (resp., chain) fragment $\mathrm{MPL}[\tau]$ (resp., $\mathrm{MCL}[\tau]$), which is obtained by constraining set variables to be evaluated over paths (resp., chains), together with their $\mathcal{P}$-variants $\mathrm{FO}_{\mathcal{P}}[\tau]$ and $\mathrm{MCL}_{\mathcal{P}}[\tau]$ (resp., $\mathrm{MPL}_{\mathcal{P}}[\tau]$)[2]. It is worth pointing out that, while free set variables in the path (resp., chain) fragments are evaluated over the set of paths (resp., chains), there are no constraints on the valuation of symbols $\mathcal{P}$ in the first-order, path, and chain fragments. As a consequence, we have that the satisfiability problem for $\mathrm{FO}_{\mathcal{P}}[\tau]$, $\mathrm{MPL}_{\mathcal{P}}[\tau]$, and $\mathrm{MCL}_{\mathcal{P}}[\tau]$ is more difficult than that for $\mathrm{FO}[\tau]$, $\mathrm{MPL}[\tau]$, and $\mathrm{MCL}[\tau]$.

To compare the various logics, we take advantage of a suitable notion of *reducibility*. We say that (the satisfiability problem for) a logic $\mathcal{L}'$ is *reducible* to (the satisfiability problem for) a logic $\mathcal{L}$, denoted $\mathcal{L}' \rightarrow \mathcal{L}$, if there exists an effective translation of $\mathcal{L}$-formulas into equi-satisfiable $\mathcal{L}'$-formulas (as a general rule, the number and types of free variables in the former formulas may not coincide with the number and types of free variables in the latter formulas). Moreover, we say that $\mathcal{L}$ and $\mathcal{L}'$ are *inter-reducible*, denoted $\mathcal{L} \rightleftarrows \mathcal{L}'$, if both $\mathcal{L}' \rightarrow \mathcal{L}$ and $\mathcal{L} \rightarrow \mathcal{L}'$. It is immediate to see that if $\mathcal{L}' \rightarrow \mathcal{L}$ and $\mathcal{L}$ is decidable (resp., $\mathcal{L}'$ is undecidable), then $\mathcal{L}'$ is decidable (resp., $\mathcal{L}$ is undecidable) as well. A well-known method to reduce the satisfiability problem for a logic $\mathcal{L}'$ to the satisfiability problem for a logic $\mathcal{L}$ is to define an *interpretation* of $\mathcal{L}'$ into $\mathcal{L}$, namely, to find (i) a mapping γ from elements in the relational structure of $\mathcal{L}'$ to elements in the relational structure of $\mathcal{L}$ and (ii) a mapping α from atomic $\mathcal{L}'$-formulas to $\mathcal{L}$-formulas with the same free variables in such a way that an $\mathcal{L}'$-formula $\phi(x_1, ..., x_m, X_1, ..., X_n)$ holds with a valuation $(c_1, ..., c_m, b_1, ..., b_n)$ if and only if the corresponding formula $\alpha(\phi(x_1, ..., x_m, X_1, ..., X_n))$ holds with a valuation $(\gamma(c_1), ..., \gamma(c_m), \gamma(b_1), ..., \gamma(b_n))$ (here the mappings α and γ are extended in the natural way to Boolean combinations and existential closures of atomic formulas and to sets b_i, respectively). It is immediate to show that if there is an interpretation of a logic $\mathcal{L}'$ into a logic $\mathcal{L}$, then $\mathcal{L}'$ is trivially reducible to $\mathcal{L}$. If there exists also a converse interpretation of $\mathcal{L}$ into $\mathcal{L}'$, we may conclude that $\mathcal{L}$ to $\mathcal{L}'$ are inter-reducible Let σ be a unary (resp., binary) relational symbol. We say that σ is *definable* in $\mathrm{MSO}[\tau]$ if there is an interpretation of $\mathrm{MSO}[\tau \cup \{\sigma\}]$, in particular, of atomic formulas of the form $\sigma(x)$ (resp., $\sigma(x, y)$), into $\mathrm{MSO}[\tau]$. Clearly, if σ is definable in $\mathrm{MSO}[\tau]$, then $\mathrm{MSO}[\tau \cup \{\sigma\}]$ and $\mathrm{MSO}[\tau]$ are inter-reducible. The notion of definability naturally transfers to any fragment of $\mathrm{MSO}[\tau]$.

The considered logics will be interpreted over different relational structures. Besides the discrete linear order $\langle \mathbb{N}, < \rangle$, we shall focus our attention on the following

2. The definitions of path and chain differ from one ω-layered structure to the other and they will be formalized later on.

ω-layered structures for time granularity. They differ from each other in several respects, but they share the notion of layer: for any $i \in \mathbb{Z}$, we denote by T_i the set $\{a_i : a \in \mathbb{N}\}$ and we call it a *layer* of the ω-layered structure.

The upward unbounded layered structure

Let $\mathcal{U} = \bigcup_{i \geq 0} T_i$. For any $k \geq 2$, the k-refinable *upward unbounded layered structure* (abbreviated UULS) is a triplet $\langle \mathcal{U}, (\downarrow_i)_{i=0}^{k-1}, < \rangle$, which intuitively represents a complete k-ary infinite tree generated from the leaves (see Figure 1). The set $\mathcal{U}$ is the domain of the structure, defined as the union of all non-negative layers, $\downarrow_i$, with $i = 0, ..., k-1$, is a projection function such that $\downarrow_i (a_0) = \perp$ for all $a \in \mathbb{N}$ and $\downarrow_i (a_b) = c_d$ if and only if $b > 0$, $d = b - 1$, and $c = a \cdot k + i$, and $<$ is the total ordering of $\mathcal{U}$ given by the *inorder* (left-root-right) visit of the tree-shaped structure. A (full) *path* over an UULS is a subset of the domain whose elements can be written as an (infinite) sequence $x_0, x_1, ...$ such that, for every $i > 0$, there exists $0 \leq j < k$ such that $x_{i-1} = \downarrow_j (x_i)$. Notice that every pair of infinite paths over an UULS may differ on a finite prefix only. A *chain* is any subset of a path.

The downward unbounded layered structure

Let $\mathcal{D} = \bigcup_{i \leq 0} T_i$. For any $k \geq 2$, the k-refinable *downward unbounded layered structure* (DULS for short) is a triplet $\langle \mathcal{D}, (\downarrow_i)_{i=0}^{k-1}, < \rangle$, which can be viewed as an infinite sequence of complete k-ary infinite trees (see Figure 2). The set $\mathcal{D}$ is the domain of the structure, defined as the union of all non-positive layers, $\downarrow_i$, with $i = 0, ..., k-1$, is a projection function such that $\downarrow_i (a_b) = c_d$ if and only if $d = b - 1$ and $c = a \cdot k + i$, and $<$ is the total ordering of $\mathcal{D}$ induced by the natural ordering on the top layer T_0 (i.e. $0_0 < 1_0 < 2_0 < ...$) and by the *preorder* (root-left-right) visit of the elements belonging to the same tree. A (full) *path* over a DULS is a subset of the domain $\mathcal{D}$ whose elements can be written as an (infinite) sequence $x_0, x_{-1}, ...$ such that, for every $i \leq 0$, there exists $0 \leq j < k$ such that $x_{i-1} = \downarrow_j (x_i)$. A *chain* is any subset of a path.

The totally unbounded layered structure

Let $\mathcal{T} = \bigcup_{i \in \mathbb{Z}} T_i$. For any $k \geq 2$, the k-refinable *totally unbounded layered structure* (TULS for short) is the merging of the k-refinable DULS and the k-refinable UULS (see Figure 3). It can be formally defined as the triplet $\langle \mathcal{T}, (\downarrow_i)_{i=0}^{k-1}, < \rangle$, where $\downarrow_i$, with $i = 0, ..., k-1$, is a projection function such that $\downarrow_i (a_b) = c_d$ if and only if $d = b - 1$ and $c = a \cdot k + i$, and $<$ is the total ordering of $\mathcal{T}$ given by the *inorder* (left-root-right) visit of the tree-shaped structure. A (full) *path* over the TULS is a subset of the domain whose elements can be written as a (bi-infinite) sequence ..., x_{-1}, x_0, x_1, ... such that, for every $i \in \mathbb{Z}$, there exists $0 \leq j < k$ such that $x_{i-1} = \downarrow_j (x_i)$. A *chain* is any subset of a path.

A $\mathcal{P}$-labeled UULS (resp., DULS, TULS) is obtained by expanding the UULS (resp., DULS, TULS) with a set $P \subseteq \mathcal{U}$ (resp., $P \subseteq \mathcal{D}$, $P \subseteq \mathcal{T}$) for each predicate in $\mathcal{P}$, which represents all elements where the predicate holds.

The MSO theory of the UULS (resp., DULS, TULS), denoted MSO$[<, (\downarrow_i)_{i=0}^{k-1}]$, is an MSO theory with equality and the binary relational symbols $<, \downarrow_0$, ..., and $\downarrow_{k-1}$. In fact, the ordering relation $<$ can be removed from the theories of the UULS and the TULS, because it can be defined in both structures by suitable $\mathrm{MSO}[(\downarrow_i)_{i=0}^{k-1}]$-formulas which use the projection functions only. On the contrary, the theory of the DULS devoid of the ordering relation $<$ is strictly less expressive than the original theory, because the ordering on the top layer T_0 cannot be defined in terms of the projection functions. Moreover, the theories of the UULS and the DULS are embeddable into the theory of the TULS expanded with the unary predicate T_0. The decidability of the MSO theories of the UULS, DULS, and TULS, possibly expanded with the predicate T_0, has been proved by reducing the underlying relational structures to suitable 'collapsed' structures. In particular, the MSO theory of the k-refinable UULS is embeddable into the MSO theory of the k-ary systolic tree, while the MSO theories of the k-refinable DULS and TULS are embeddable into the MSO theory of the infinite complete k-ary tree [MON 99, MON 00a, MON 02, MON 04].

THEOREM 2. — *The satisfiability problem for* $\mathrm{MSO}[<, (\downarrow_i)_{i=0}^{k-1}]$ *(which coincides with* $\mathrm{MSO}_{\mathcal{P}}[<, (\downarrow_i)_{i=0}^{k-1}]$*) over the k-refinable DULS, UULS, and TULS is (nonelementarily) decidable.*

Figure 4 summarizes the relationships between the considered MSO logics and their fragments induced by reducibility (an arrow from $\mathcal{L}'$ to $\mathcal{L}$ stands for $\mathcal{L}' \rightarrow \mathcal{L}$). From Theorem 2 it follows that the satisfiability problem for all logics in Figure 4, when interpreted over the UULS, DULS, and TULS, are decidable.

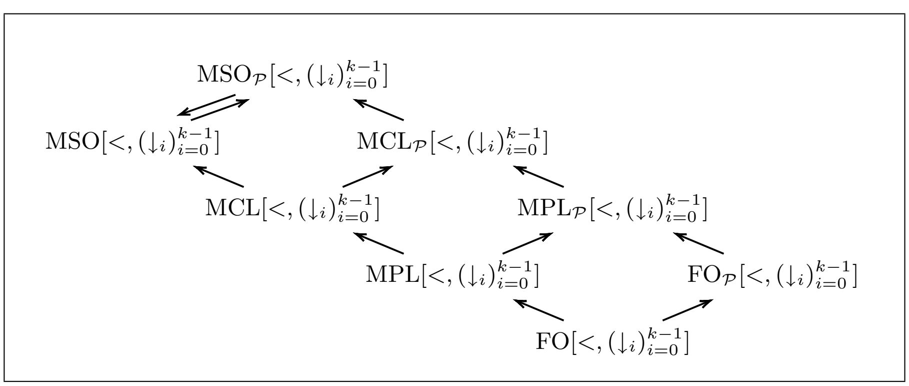

Figure 4. *A hierarchy of logics over ω-layered structures induced by reducibility*

3. On the relationships between $\mathrm{MSO}[<]$, $\mathrm{MSO}[<, \downarrow_0, \downarrow_1]$, and $\mathrm{MSO}[<, flip]$

In this section, we discuss reducibility relationships between logics over (an expansion of) $\langle \mathbb{N}, < \rangle$ and logics over the ULLS, focussing our attention on reductions obtained via interpretation.

It is immediate to define an interpretation of $\mathrm{MSO}[<]$ over $\langle \mathbb{N}, < \rangle$ into $\mathrm{MSO}[<, \downarrow_0, \downarrow_1]$ over the binary ULLS. Here, we show that $\mathrm{MSO}[<, \downarrow_0, \downarrow_1]$ is reducible (via interpretation) to $\mathrm{MSO}[<, flip]$, which is the proper extension of $\mathrm{MSO}[<]$ with the binary relation symbol $flip$ [MON 99, MON 00b].

In [MON 00a], Montanari et al. describe in detail how the basic temporal operators for time granularity, namely, the displacement, contextualization, and projection operators, can be defined in $\mathrm{MSO}[<, \downarrow_0, \downarrow_1]$ over the UULS. As an example, we report the definition of the unary predicate Δ_0 which holds at the origin of each layer. The predicate Δ_0 is interpreted as the set of all and only the elements belonging to the leftmost branch of an ULLS, which is defined as the least set containing the element 0_0 and all its ancestors $0_1, 0_2, \ldots$. This predicate can be defined in $\mathrm{MSO}[<, \downarrow_0, \downarrow_1]$ as follows. Given an MSO formula $\phi(X)$, with a free set variable X, let $\mu(\phi(X))(x)$ be the following formula, with a free individual variable x:

$$\exists X \ (x \in X \ \wedge \ \phi(X) \ \wedge \ \forall Y \ (\phi(Y) \rightarrow \forall y (y \in X \rightarrow y \in Y))).$$

$\mu(\phi(X))(x)$ evaluates to true if and only if the valuation for x belongs to the smallest valuation for X for which $\phi(X)$ holds true. Using the operator μ, $\Delta_0(x)$ can be expressed as follows:

$$\mu \left(0_0 \in X \ \wedge \ \forall y, z \left((z \in X \wedge \ \downarrow_0 (y) = z) \rightarrow y \in X\right)\right)(x),$$

where $0_0 \in X$ is a shorthand for $\exists y \ (y \in X \ \wedge \ \forall z \ (y \leq z))$. It is easy to verify that such a formula captures the smallest valuation for X which contains 0_0 and it is closed parent-wise. This shows that Δ_0 is definable in $\mathrm{MSO}[<, \downarrow_0, \downarrow_1]$.

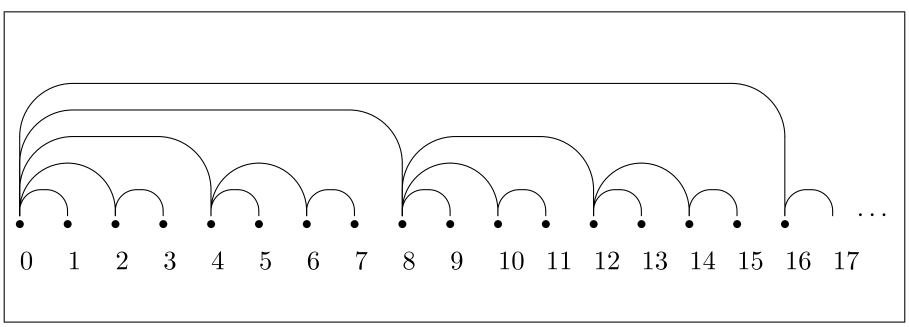

Figure 5. *The structure of the function $flip$*

The language of $\mathrm{MSO}[<, flip]$ is defined in the standard way. The domain of the underlying relational structure is the set of natural numbers $\mathbb{N}$; moreover, the relational symbol $<$ is interpreted as the usual ordering over $\mathbb{N}$, while the relational symbol $flip$ is interpreted as a unary function, which, for any natural number $x > 0$, returns the natural number $x - x'$, where x' is the least power of 2, with a non-null coefficient, that occurs in the binary representation of x.

DEFINITION 3 (THE FUNCTION $flip$). — *The function $flip : \mathbb{N}^+ \to \mathbb{N}$ is defined as follows: for all $x \in \mathbb{N}^+$,*

$$flip(x) = y \textit{ iff } x = \sum_{j=0}^{n} 2^{i_j}, \textit{ with } i_n > i_{n-1} > \ldots > i_0 \geq 0, \textit{ and } y = x - 2^{i_0}.$$

The function $flip$ is not defined for $x = 0$; however, totality can be recovered by extending it with $flip(0) = 0$. (Notice that $flip(x) < x$, for all $x \in \mathbb{N}^+$, and $flip(x) \leq x$, for all $x \in \mathbb{N}$. Later, we will often use these properties of $flip$ to simplify definitions.) Furthermore, it is useful to add a maximum element ∞ to $\mathbb{N}$, with $flip(\infty) = 0$. A graphical representation of the function $flip$ is given in Figure 5.

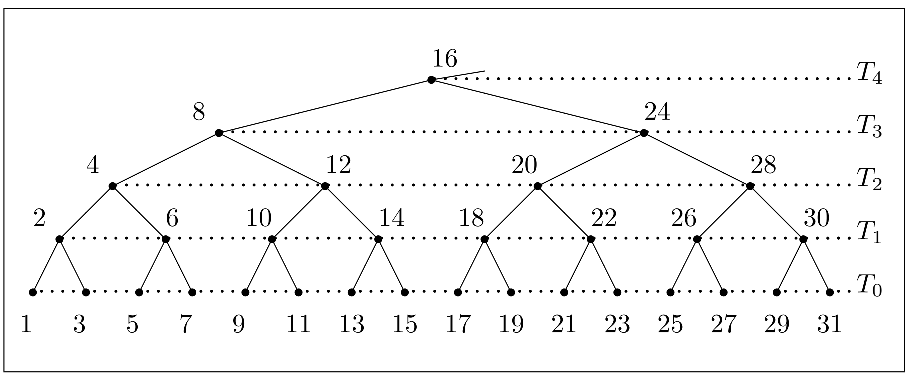

Figure 6. *The concrete* 2*-refinable UULS*

An interpretation of $\mathrm{MSO}[<, \downarrow_0, \downarrow_1]$ into $\mathrm{MSO}[<, flip]$ can be defined as follows [MON 99]. First, it is possible to rename each node a_b of the 2-refinable UULS by a positive natural number $\gamma(a_b) = 2^b + a2^{b+1}$. The resulting structure is called *concrete* 2-refinable UULS and it can be viewed as the (discrete) linear order $\langle \mathbb{N}^+, < \rangle$ expanded with two functions $\downarrow_0$ and $\downarrow_1$ such that, for every $x = 2^b + a2^{b+1} \in \mathbb{N}^+$, $\downarrow_0 (x) = x - 2^{b-1}$ and $\downarrow_1 (x) = x + 2^{b-1}$. A fragment of this concrete structure is depicted in Figure 6. Notice that all odd numbers are associated with layer T_0, while even numbers are distributed over the remaining layers. Notice also that the labeling of the concrete structure does not include the number 0. For the sake of convenience, we will consider 0 as the image of the first node of an imaginary additional finest layer, whose remaining nodes have no corresponding number in $\mathbb{N}$ (notice that in such a way the node corresponding to 0 turns out to be the left son of the node corresponding to 1). Since the addition/removal of a (definable) node in a structure preserves the expressive power of the corresponding logic, we do not need to explicitly care about such an element in the following proofs.

The binary relations $\downarrow_0$ and $\downarrow_1$ of the concrete 2-refinable UULS can be defined neither in FO[<] nor in $\mathrm{MSO}[<]$ (this can be proved by showing that the relation $\downarrow_0$ restricted to the elements of the leftmost branch coincides with the relation $\{(2x, x) : x \in \mathbb{N}^+\}$). This implies that $\mathrm{MSO}[<, \downarrow_0, \downarrow_1]$ (resp. $\mathrm{FO}[<, \downarrow_0, \downarrow_1]$) is strictly more expressive than $\mathrm{MSO}[<]$ (resp. FO[<]). However, both relations $\downarrow_0$ and $\downarrow_1$ can be defined in terms of the function $flip$ as shown by Theorem 4 below. As a preliminary step, notice that for every even natural number x, we have:

$\downarrow_0 (x) = max\{y : y < x, flip(y) = flip(x)\}$ and $\downarrow_1 (x) = max\{y : flip(y) = x\}$.

Such a correspondence can be translated into suitable first-order logical formulas, thus implying that both relations $\downarrow_0$ and $\downarrow_1$ are definable in $\mathrm{FO}[<, flip]$.

THEOREM 4. — $\mathrm{MSO}[<, \downarrow_0, \downarrow_1]$ *(resp.* $\mathrm{FO}[<, \downarrow_0, \downarrow_1]$*) over the (concrete) UULS can be reduced via interpretation to* $\mathrm{MSO}[<, flip]$ *(resp.* $\mathrm{FO}[<, flip]$*) over* $\langle \mathbb{N}, < \rangle$ *expanded with* $flip$*.*

PROOF. — In order to prove the claim it suffices to provide a mapping α of the two atomic formulas $\downarrow_0 (x, y)$ and $\downarrow_1 (x, y)$ into suitable $\mathrm{FO}[<, flip]$-formulas $\alpha(\downarrow_0 (x, y))$ and $\alpha(\downarrow_1 (x, y))$. Such a mapping is defined as follows:

$$
\begin{aligned}
\alpha(\downarrow_0 (x, y)) \quad &:= \quad y < x \ \wedge \ flip(y) = flip(x) \ \wedge \\
&\qquad \forall z \, ((z < x \ \wedge \ flip(z) = flip(x)) \ \rightarrow \ (z = y \ \vee \ z < y)); \\
\alpha(\downarrow_1 (x, y)) \quad &:= \quad flip(y) = x \ \wedge \ \forall z \, (flip(z) = x \ \rightarrow \ (z = y \ \vee \ z < y)).
\end{aligned}
$$

■

In [MON 00b], Monti and Peron show that the satisfiability problem for $\mathrm{MSO}[<, flip]$ is (non-elementarily) decidable. By Theorem 4, from such a result it immediately follows that the satisfiability problem for $\mathrm{MSO}[<, \downarrow_0, \downarrow_1]$ is decidable. The same argument can be applied to the FO fragments of the considered logics.

We conclude the section by providing an interpretation of $\mathrm{MSO}[<, flip]$ (resp. $\mathrm{FO}[<, flip]$) into the MSO (resp. FO) logic of a suitable expansion of the 2-refinable (concrete) UULS. To this end, we consider the reflexive and transitive closure of $\downarrow_0$, denoted by $\downarrow_0^*$. The relation $flip$ can be defined in terms of $\downarrow_0^*$ and $\downarrow_1$ as follows:

$$flip(x) = y \quad \text{iff} \quad (\downarrow_1 (y), x) \in \downarrow_0^* \ \text{ or } \ (y = 0 \ \wedge \ (x, 0) \in \downarrow_0^*).$$

From this, it immediately follows that the relation $flip$ is definable in $\mathrm{FO}[<, \downarrow_0^*, \downarrow_1]$.

THEOREM 5. — $\mathrm{MSO}[<, flip]$ *(resp.* $\mathrm{FO}[<, flip]$*) over* $\langle \mathbb{N}, < \rangle$ *expanded with the* $flip$ *can be reduced via interpretation to* $\mathrm{MSO}[<, \downarrow_0^*, \downarrow_1]$ *(resp.* $\mathrm{FO}[<, \downarrow_0^*, \downarrow_1]$*) over the (concrete) UULS.*

PROOF. — We simply need to translate the atomic $\mathrm{FO}[<, flip]$-formula $flip(x, y)$ (which holds if and only if $y = flip(x)$) into an equivalent $\mathrm{FO}[<, \downarrow_0^*, \downarrow_1]$-formula $\beta(flip(x, y))$. This can be done by defining the mapping β as follows:

$$\beta(flip(x, y)) \quad := \quad \exists z \, (\downarrow_1 (y, z) \wedge \ \downarrow_0^* (z, x)) \ \vee \ (y = 0 \wedge \ \downarrow_0^* (x, y)),$$

where $y = 0$ is a shorthand for $\forall z\ (y = z \ \vee\ y < z)$. ∎

As for the relationships between $\downarrow_0$ and $\downarrow_0^*$, we have that $\downarrow_0$ can be defined in $\mathrm{FO}[<, \downarrow_0^*, \downarrow_1]$:

$$\downarrow_0 (x) = y \quad \text{iff} \quad \downarrow_0^* (x, y) \ \wedge\ \neg \exists z\ (y < z \wedge\ \downarrow_0^* (x, z)).$$

This allows us to conclude that $\mathrm{FO}[<, \downarrow_0, \downarrow_1]$ (resp. $\mathrm{MSO}[<, \downarrow_0, \downarrow_1]$) can be reduced via interpretation to $\mathrm{FO}[<, \downarrow_0^*, \downarrow_1]$ (resp. $\mathrm{MSO}[<, \downarrow_0^*, \downarrow_1]$). Moreover, one can easily show that $\mathrm{FO}[<, \downarrow_0^*, \downarrow_1]$ cannot be reduced to $\mathrm{FO}[<, \downarrow_0, \downarrow_1]$ (the unary predicate $\{2^n : n \in \mathbb{N}\}$ is definable in $\mathrm{FO}[<, \downarrow_0^*, \downarrow_1]$ but not in $\mathrm{FO}[<, \downarrow_0, \downarrow_1]$). On the other hand, since the reflexive and transitive closure of a binary predicate is always definable in MSO logic, we have that $\mathrm{MSO}[<, \downarrow_0^*, \downarrow_1]$ and $\mathrm{MSO}[<, \downarrow_0, \downarrow_1]$ are inter-reducible.

The reducibility relationships among the various FO and MSO logics over (an expansion of) $\langle \mathbb{N}, < \rangle$ and (an expansion of) the ULLS are summarized in Figure 7, where a bold arrow from a logic $\mathcal{L}'$ to a logic $\mathcal{L}$ means that $\mathcal{L}'$ can be reduced via interpretation to $\mathcal{L}$.

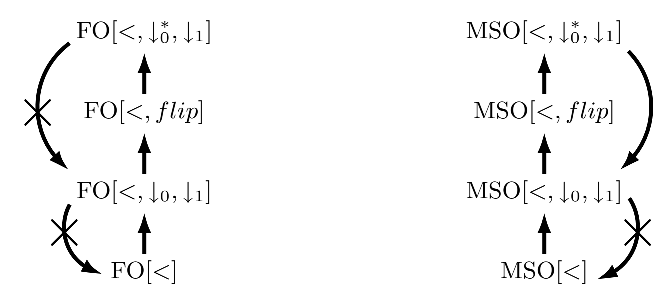

Figure 7. *Reducibility relationships between FO and MSO logics over (an expansion of) $\langle \mathbb{N}, < \rangle$ and (an expansion of) the binary ULLS*

4. On the relationships between $\mathrm{FO}[<, \downarrow_0, \downarrow_1, Path^<, D_0]$ and $\mathrm{MSO}[<]$

In this section, we show that the MSO logic of $\langle \mathbb{N}, < \rangle$ (abbreviated $\mathrm{MSO}[<]$) and the FO logic of the expanded 2-refinable UULS $(\mathcal{U}, <, \downarrow_0, \downarrow_1, Path^<, D_0)$ are inter-reducible. The predicate $Path^<$ subsumes both the equi-level predicate T and the ancestor predicate $\downarrow^\star$, while D_0 holds at all and only the elements belonging to the leftmost branch of the tree. Such a result defines the precise relationship that holds between the logic $\mathrm{MSO}[<]$ over the flat structure of natural numbers and the logic $\mathrm{FO}[<, \downarrow_0, \downarrow_1, Path^<, D_0]$ over the UULS, showing the rather surprising fact that the satisfiability problem for $\mathrm{MSO}[<]$ is reducible to the satisfiability problem for a suitable FO logic over the UULS. Moreover, the opposite reduction from $\mathrm{FO}[<, \downarrow_0, \downarrow_1, Path^<, D_0]$ to $\mathrm{MSO}[<]$ has a nontrivial impact from a practical point of view, since it allows one to map verification problems for the UULS to verification problems for MSO$[<]$, thus making it possible to exploit the wide spectrum of techniques available for that logic.

We start by defining the predicate $Path^<$. We have that $Path^<(x, y, z, w)$ holds if and only if $T(x, z)$ (x and z belong to the same layer), $T(y, w)$ (y and w belong to the same layer), and there exist two finite downward paths, one from x to y and the other from z to w, such that, for each right projection in the path from x to y, there exists a right projection in the path from z to w. More formally, we require that

– $T(x, z)$ and $T(y, w)$;

– there are two paths $c_0, \ldots, c_n$ and $b_0, \ldots, b_n$ such that $x = c_0$, $y = c_n$, $z = b_0$, $w = b_n$, $c_{i+1} = \downarrow_{i_c} (c_i)$, and $b_{i+1} = \downarrow_{i_b} (b_i)$, with $i_b, i_c \in \{0, 1\}$ for $0 \leq i \leq n-1$;

– for all $0 \leq i \leq n-1$, $\downarrow_{i_c} = \downarrow_1$ implies $\downarrow_{i_b} = \downarrow_1$.

The predicate $Path^<(x, y, z, w)$ subsumes the equi-level predicate $T(x, y)$, since $T(x, y)$ is equivalent to $Path^<(x, x, y, y)$. Moreover, it also subsumes the ancestor predicate $\downarrow^\star (x, y)$. By definition, $\downarrow^\star (x, y)$ holds true if and only if either x is equal to y or x is an ancestor of y, that is, there exists a finite path $c_0, \ldots, c_n$ such that $c_0 = x$, $c_n = y$, and $c_{i+1} = \downarrow_i (c_i)$, for $0 \leq i \leq n-1$, and thus $\downarrow^\star (x, y)$ is equivalent to $Path^<(x, y, x, y)$ (in the following we will often use $\downarrow^\star (x, y)$ as a shorthand for $Path^<(x, y, x, y)$).

THEOREM 6. — FO$[<, \downarrow_0, \downarrow_1, Path^<, D_0]$ *over the expanded* 2*-refinable UULS and* MSO$[<]$ *over* $\langle \mathbb{N}, < \rangle$ *are inter-reducible.*

PROOF. — We first prove that MSO$[<]$ can be reduced to FO$[<, \downarrow_0, \downarrow_1, Path^<, D_0]$. As a first step, we replace MSO$[<]$ with *Weak* MSO$[<]$ (WMSO$[<]$ for short), where second-order quantification refers to finite sets only. By the well-known McNaughton Theorem (see, for instance, [**?**]), MSO$[<]$ and WMSO$[<]$ have the same expressive power, and thus such a replacement is legitimate. Moreover, we replace WMSO$[<]$ with the simpler, but equivalent, formalism WMSO$_0[\subseteq, Succ]$ where only second-order variables occur and atomic formulas are of the forms $X \subseteq Y$ (X is a subset of Y) and $Succ(X, Y)$ (X and Y are the singletons $\{x\}$ and $\{y\}$, respectively, and $y = x + 1$).

The reduction is based on a suitable encoding of (finite) sets of natural numbers into elements of the concrete 2-refinable UULS. More precisely, any second-order variable X of WMSO$_0[\subseteq, Succ]$ is replaced with a first-order variable x of FO$[<, \downarrow_0, \downarrow_1, Path^<, D_0]$ and any interpretation $\nu(X)$ of X is mapped into an interpretation $\mu_\nu(x)$ of x as follows:

– if $\nu(X) = \emptyset$, then $\mu_\nu(x)$ is the origin of the UULS, that is, the lowest element to the left;

– if $\nu(X) = \{n_0, n_1, \ldots, n_s\}$, then $\mu_\nu(x)$ is the element a_b of the UULS such that $2^b + a2^{b+1} = 2^{n_s} + \ldots + 2^{n_1} + 2^{n_0}$.

Since in $\text{WMSO}_0[\subseteq, Succ]$ the interpretation $\nu(X)$ of any second-order variable X is finite, we have that rule 2 is effective. An intuitive account of the mapping ν can be given in terms of the concrete 2-refinable UULS depicted in Figure 6: the set $\nu(X)$ is the set of positions of the non-zero coefficients of the binary representation of $\mu_\nu(x)$.

Later in the proof, we will take advantage of the following interpretation of the set $\nu(X)$ as a path over the concrete UULS. First of all, since in $\text{WMSO}_0[\subseteq, Succ]$ the interpretation of any set variable is finite, $\nu(X)$ has not only a least element $min(\nu(X))$, but also a greatest element $max(\nu(X))$. We associate $\nu(X)$ with the path from the origin of the layer $T_{max(\nu(X))}$ to the element $\mu_\nu(x)$, belonging to the layer $T_{min(\nu(X))}$. Such a path provides an encoding of the elements of $\nu(X)$ as follows: $max(\nu(X))$, that is, the index of the layer of the first element in the path, and $min(\nu(X))$, that is, the index of the layer of the last element in the path, belong to $\nu(X)$; moreover, if the element a_b of the UULS, with $min(\nu(X)) < b \leq max(\nu(X))$, belongs to the path, then $\downarrow_1 (a_{b-1})$ belongs to the path if (and only if) $b - 1 \in \nu(X)$ and $\downarrow_0 (a_{b-1})$ belongs to the path if (and only if) $b - 1 \notin \nu(X)$.

On the ground of the above-defined correspondence, we can translate every of $\text{WMSO}_0[\subseteq, Succ]$-formula ϕ into an $\text{FO}[<, \downarrow_0, \downarrow_1, Path^<, D_0]$-formula $\alpha(\phi)$, where the mapping α is inductively defined as follows:

$$
\begin{aligned}
\alpha(Succ(X,Y)) \quad &:= \quad D_0(x) \wedge D_0(y) \wedge \downarrow_0 (y) = x;\\
\alpha(X \subseteq Y) \quad &:= \quad x = y \vee (x < y \wedge \exists z, w\ (D_0(z) \wedge (Path^<(z,x,w,y) \vee\\
&\qquad \exists h,k\ (Path^<(z,x,w,k) \wedge h = \downarrow_1 (k) \wedge \downarrow^\star (h,y)))));\\
\alpha(\phi \ \wedge\ \psi) \quad &:= \quad \alpha(\phi) \wedge \alpha(\psi);\\
\alpha(\neg\phi) \quad &:= \quad \neg\alpha(\phi);\\
\alpha(\exists X\ \phi) \quad &:= \quad \exists x \alpha(\phi).
\end{aligned}
$$

The rules for atomic formulas can be explained by taking into account the relationship that holds between interpretations of set variables in $\text{WMSO}_0[\subseteq, Succ]$ and interpretations of the corresponding individual variables in $\text{FO}[<, \downarrow_0, \downarrow_1, Path^<, D_0]$ as well as the structure of the concrete 2-refinable UULS. As for the formula $Succ(X,Y)$, it suffices to notice that singletons are mapped into elements which are powers of two, and thus belong to the leftmost branch of the concrete UULS, and that the successor relation can be directly captured by the left projection. The translation of the formula $X \subseteq Y$ is more involved. The case in which $X = Y$ is trivial, and thus we concentrate our attention on the case $X \subset Y$. As anticipated, we take advantage of the interpretation of $\nu(X)$ and $\nu(Y)$ as paths over the concrete UULS. In order to guarantee that $X \subset Y$ we have to check that at every layer T_i, with $min(\nu(X)) < i \leq max(\nu(X))$, if the path associated with $\nu(X)$ follows a right projection, then the path associated with $\nu(Y)$ follows a right projection as well (notice that, in general, the path associated with $\nu(Y)$ may be longer than the one associated with $\nu(X)$). This can be ensured by exploiting predicate $Path^<$.

From the given translation of $\mathrm{WMSO}_0[\subseteq, Succ]$ into $\mathrm{FO}[<, \downarrow_0, \downarrow_1, Path^<, D_0]$, and the correspondence between the interpretation of set variables in $\mathrm{WMSO}_0[\subseteq, Succ]$ and the interpretation of the corresponding individual variables in $\mathrm{FO}[<, \downarrow_0, \downarrow_1, Path^<, D_0]$, it is easy to show that a $\mathrm{WMSO}_0[\subseteq, Succ]$-formula ϕ is satisfiable, with an interpretation ν, if and only if $\alpha(\phi)$ is satisfiable, with the interpretation μ_ν.

Consider now the opposite reduction from the logic $\mathrm{FO}[<, \downarrow_0, \downarrow_1, Path^<, D_0]$ to the logic $\mathrm{MSO}[<]$. As before, we define an injective function that maps each individual variable x of $\mathrm{FO}[<, \downarrow_0, \downarrow_1, Path^<, D_0]$ into a set variable X of $\mathrm{MSO}[<]$. The encoding of the elements of the UULS into natural numbers is exactly the reverse of the previously-defined encoding, which induces, for an interpretation μ over the UULS, an interpretation ν_μ over the natural numbers. A formula ϕ of $\mathrm{FO}[<, \downarrow_0, \downarrow_1, Path^<, D_0]$ is mapped into a formula $\beta(\phi)$ of $\mathrm{FO}[<]$ by a function β inductively defined as follows:

$$
\begin{aligned}
\beta(x=y) &:= \overline{X}=\overline{Y};\\
\beta(D_0(x)) &:= \exists z\ (z\in\overline{X}\ \wedge\ \forall h\ (h\in\overline{X}\rightarrow h=z));\\
\beta(x<y) &:= \exists z\ (z\in\overline{Y}\ \wedge\ z\notin\overline{X}\ \wedge\\
&\qquad \forall w\ ((w\in\overline{X}\wedge w\notin\overline{Y})\rightarrow w<z));\\
\beta(Path^{<}(x,y,z,w)) &:= \neg\exists z\ (z\in\overline{X}\vee z\in\overline{Y}\ \vee z\in\overline{Z}\vee z\in\overline{W})\ \vee\\
&\quad \exists h,k\ (h\in\overline{X}\ \wedge\ h\in\overline{Z}\ \wedge k\in\overline{Y}\ \wedge\ k\in\overline{W}\ \wedge\\
&\qquad\quad \forall v\ (v\in\overline{X}\rightarrow h\le v\wedge v\in\overline{Z}\rightarrow h\le v)\ \wedge\\
&\qquad\quad \forall v\ (v\in\overline{Y}\rightarrow k\le v\wedge v\in\overline{W}\rightarrow k\le v)\ \wedge\\
&\qquad\quad \forall v\ ((v\in\overline{X}\wedge v>h)\rightarrow v\in\overline{Y}\wedge\\
&\qquad\qquad (v\in\overline{Y}\wedge v>h)\rightarrow v\in\overline{X})\ \wedge\\
&\qquad\quad \forall v\ ((v\in\overline{Z}\wedge v>k)\rightarrow v\in\overline{W}\wedge\\
&\qquad\qquad (v\in\overline{W}\wedge v>k)\rightarrow v\in\overline{Z})\ \wedge\\
&\qquad\quad \forall v\ ((v<h\wedge k\le v\wedge v\in\overline{Y})\rightarrow v\in\overline{W}));\\
\beta(\downarrow_0(x)=y) &:= \exists z\ (z\in\overline{X}\ \wedge\ z\notin\overline{Y}\ \wedge\\
&\qquad \forall w\ (w\in\overline{X}\rightarrow z\le w)\ \wedge\\
&\qquad \forall w\ ((w\in\overline{X}\wedge z<w)\rightarrow w\in\overline{Y}\ \wedge\\
&\qquad\quad (w\in\overline{Y}\wedge z\le w)\rightarrow w\in\overline{X}\ \wedge\\
&\qquad\quad (w\in\overline{Y}\wedge w<z)\rightarrow z=w+1)\ \wedge\\
&\qquad \exists w\ (w\in\overline{Y}\wedge w<z));\\
\beta(\downarrow_1(x)=y) &:= \exists z\ (z\in\overline{X}\ \wedge\\
&\qquad \forall w\ (w\in\overline{X}\rightarrow z\le w)\ \wedge\\
&\qquad \forall w\ ((w\in\overline{X}\wedge z\le w)\rightarrow w\in\overline{Y}\wedge\\
&\qquad\quad (w\in\overline{Y}\wedge z\le w)\rightarrow w\in\overline{X}\wedge\\
&\qquad\quad (w\in\overline{Y}\wedge w<z)\rightarrow z=w+1)\ \wedge\\
&\qquad \exists w\ (w\in\overline{Y}\wedge w<z));\\
\beta(\phi\wedge\psi) &:= \beta(\phi)\ \wedge\ \beta(\psi);\\
\beta(\neg\phi) &:= \neg\beta(\phi);\\
\beta(\exists x\ \phi) &:= \exists\overline{X}\ \beta(\phi).
\end{aligned}
$$

The translation β can be explained by referring to the concrete UULS structure over natural numbers. The predicate $D_0(x)$ holds if x is interpreted over a power of two, that is, if $\mu_\nu(X)$ is a singleton. As for the predicate $Path^<$, we have that the elements x and z (resp. y and w) belong to the same layer if the corresponding sets X and Z (resp. Y and W) have the same least element. Moreover, the predicate $\downarrow^\star (x, y)$ holds if the path from the leftmost branch to x, described by the set X, is a prefix of the path from the leftmost branch to y, described by Y. Finally, the translation of the projection functions $\downarrow_0$ and $\downarrow_1$ exploits the fact that, if $x = 2^{k_n} + 2^{k_{n-1}} + \ldots + 2^{k_0}$, with $k_n > k_{n-1} > \ldots > k_0 > 0$, then $\downarrow_0 (x) = y$, with $y = x - 2^{k_0} + 2^{k_0 - 1}$, and $\downarrow_1 (x) = y$, with $y = x + 2^{k_0} + 2^{k_0-1}$. ■

5. On the relationships between MCL$[<, \downarrow_0, \downarrow_1, T_0, T]$, MCL$[<, \downarrow_0, \downarrow_1, T_0, D]$, and MSO$[<]$

In this section, we explore the relationships between MSO$[<]$ and chain fragments of MSO logics interpreted over the TULS expanded with T_0 (layer 0 predicate) and either T (equi-level predicate) or D (equi-column predicate). As already pointed out, T allows one to check whether two given elements of the TULS belong to the same layer, while D allows one to check whether two given elements are at the same distance from the origin of the layer they belong to. Formally, we can define T and D as follows:

$$\begin{array}{lcl} T & := & \{(a_b, c_b) : a \in \mathbb{N}, c \in \mathbb{N}, b \in \mathbb{Z}\}; \\ D & := & \{(a_b, a_d) : a \in \mathbb{N}, b \in \mathbb{Z}, d \in \mathbb{Z}\}. \end{array}$$

The reductions we are going to describe will allow us to conclude that the satisfiability problem for the chain fragment of MSO logic interpreted over the expanded TULS $\langle \mathcal{T}, (\downarrow_i)_{i=0}^{k-1}, <, T_0, T\rangle$ (resp., $\langle \mathcal{T}, (\downarrow_i)_{i=0}^{k-1}, <, T_0, D\rangle$) is decidable. Such results have been originally provided in [PUP 06] and they are partly based on a proof method introduced by Thomas in [**?**], which makes it possible to reduce the chain fragment of an MSO logic interpreted over a tree-shaped structure into an MSO logic over the discrete linear structure $\langle \mathbb{N}, <\rangle$. As usual, for the sake of simplicity, we restrict our attention to the 2-refinable TULS.

THEOREM 7. — MCL$[<, \downarrow_0, \downarrow_1, T_0, T]$ *over the TULS and* MSO$[<]$ *over* $\langle \mathbb{N}, <\rangle$ *are inter-reducible.*

PROOF. — We show that the logic MSO$[<]$ over the natural numbers is reducible to the logic MCL$[<, \downarrow_0, \downarrow_1, T_0, T]$ over the TULS. We first replace MSO$[<]$ with the equivalent logic MSO$_0[\subseteq, Succ]$, where only second-order variables occur and atomic formulas are of the forms $X \subseteq Y$ and $Succ(X, Y)$. We denote by D_0 the unary predicate that consists of all and only the elements belonging to the leftmost upward branch, that is, the portion of the leftmost branch from level 0 upward, of the 2-refinable TULS. Such a predicate can be easily defined by a formula in the chain fragment of MSO logic over the TULS. We can translate any MSO$_0[\subseteq, Succ]$-formula

ϕ into an equi-satisfiable MCL$[<, \downarrow_0, \downarrow_1, T_0, T]$-formula $\alpha(\phi)$ by constraining every second-order variable to be interpreted with elements from D_0. Formally, the mapping α is inductively defined as follows:

$$\begin{array}{rcl}\alpha(X \subseteq Y) & := & Y \subseteq D_0 \wedge X \subseteq Y; \\ \alpha(Succ(X,Y)) & := & X \subseteq D_0 \wedge \forall x, y(x \in X \wedge y \in Y \rightarrow x = \downarrow_0 (y)); \\ \alpha(\phi \wedge \psi) & := & \alpha(\phi) \wedge \alpha(\psi); \\ \alpha(\neg\phi) & := & \neg\alpha(\phi); \\ \alpha(\exists X \phi) & := & \exists X\ \alpha(\phi).\end{array}$$

Since D_0 is definable in MCL$[<, \downarrow_0, \downarrow_1, T_0, T]$, we have that MSO$_0[\subseteq, Succ]$ (and hence MSO$[<]$) is reducible to MCL$[<, \downarrow_0, \downarrow_1, T_0, T]$.

As for the opposite reduction, we must show how to transform any $\mathrm{MCL}[<, \downarrow_0, \downarrow_1, T_0, T]$-formula into an equi-satisfiable MSO$[<]$-formula. We define such a translation in two steps. We first translate any MCL$[<, \downarrow_0, \downarrow_1, T_0, T]$-formula into an equi-satisfiable $\mathrm{MSO}[<]$-formula interpreted over the bi-infinite linear structure $\langle \mathbb{Z}, < \rangle$. Then, we exploit standard logical constructions to map the latter formula into an equi-satisfiable MSO$[<]$-formula over $\langle \mathbb{N}, < \rangle$. As for the first step, we encode chain variables with suitable pairs of second-order variables and then we give rules to rewrite atomic formulas. Notice that the ordering $<$ of the TULS can be easily defined by a formula in the chain fragment of its MSO logic. Moreover, we can assume, without loss of generality, that second-order variables of MCL$[<, \downarrow_0, \downarrow_1, T_0, T]$-formulas are interpreted by *non-empty* chains. Therefore, we can restrict ourselves to the equivalent setup of MCL$[<, \downarrow_0, \downarrow_1, T_0, T]$ where second-order variables are instantiated by non-empty chains and atomic formulas are of the forms $X \subseteq Y$ (chain X is included in chain Y), $\downarrow_i (X, Y)$ (X and Y are the singletons $\{x\}$ and $\{y\}$, respectively, and $y = \downarrow_i (x)$), $T_0(X)$ (X is the singleton $\{x\}$, with $x \in T_0$), and $T(X, Y)$ (X and Y are the singletons $\{x\}$ and $\{y\}$, respectively, and x and y belong to the same layer).

The treatment of chain variables is based on the observation that, for every non-empty chain C over the TULS and for every $b \in \mathbb{Z}$, there exists at most one $a \in \mathbb{N}$ such that a_b is an element of C. Accordingly, any non-empty chain C can be encoded into two subsets Z_C and W_C of $\mathbb{Z}$ as follows. We say that $P \subseteq \mathcal{T}$ is a *cover* of a non-empty chain C if P is a maximal path including C, namely, if $C \subseteq P$ and for every $b \in \mathbb{Z}$, there exists exactly one $a \in \mathbb{N}$ such that $a_b \in P$. We denote by P_C the *leftmost* cover of C, that is, the (unique) cover P_C such that, whenever b is the least integer for which there is $a \in \mathbb{N}$ satisfying $a_b \in C$, then every descendant of a_b along P_C is of the form c_d, with $c = 2^{b-d}a$ and $d \leq b$. Then, we define Z_C and W_C in such a way that, for every $b \in \mathbb{Z}$,

– $b \in Z_C$ iff there is a (unique) *odd* index $a \in \mathbb{N}$ such that $a_b \in P_C$ (namely, a_b is a $\downarrow_1$-successor in the path P_C);

– $b \in W_C$ iff there is a (unique) index $a \in \mathbb{N}$ such that $a_b \in C$ (namely, C intersects the layer T_b).

Intuitively, Z_C represents those layers which are reached by right-hand side projections along the path P_C, while W_C selects only those layers which intersect the chain C. Notice that the encoding (Z_C, W_C) determines in an unambiguous way the nonempty chain C. An encoding of the above construction in the logic can be obtained as follows. First of all, for every chain variable X, we introduce two set variables Z_X and W_X (to be instantiated by sets of integers). Then, we map any MCL$[<, \downarrow_0, \downarrow_1, T_0, T]$-formula ϕ into an equi-satisfiable MSO$[<]$-formula $\beta(\phi)$ by means of the following sequence of steps. As a preliminary step, we replace MCL$[<, \downarrow_0, \downarrow_1, T_0, T]$ with the equivalent logic MCL$_0[\subseteq, \downarrow_0, \downarrow_1, T_0, T]$. Moreover, without loss of generality, we existentially close every free set variable occurring in the formula ϕ in such a way that we may restrict our attention to sentences. The mapping β is inductively defined as follows (for the sake of readability, we take advantage of various obvious shorthands):

$$
\begin{array}{rcl}
\beta(X \subseteq Y) & := & W_X \subseteq W_Y \ \wedge (Z_X = Z_Y \vee \\
& & \quad \exists w\ (w \in W_X \ \wedge\ \forall w'\ (w' \in W_X \rightarrow w' \geq w) \wedge \\
& & \qquad \forall z\ (z \geq w \rightarrow (z \in Z_X \leftrightarrow z \in Z_y)))); \\
\beta(\downarrow_0 (X, Y)) & := & \exists w\ (Z_X = Z_Y \wedge W_X = \{w\} \wedge W_Y = \{w-1\}); \\
\beta(\downarrow_1 (X, Y)) & := & \exists w\ (Z_X \cup \{w-1\} = Z_Y \wedge W_X = \{w\} \wedge W_Y = \{w-1\}); \\
\beta(T_0(X)) & := & W_X = \{0\}; \\
\beta(T(X, Y)) & := & \exists w\ (W_X = W_Y = \{w\}); \\
\beta(\phi \ \wedge\ \psi) & := & \beta(\phi) \ \wedge\ \beta(\psi); \\
\beta(\neg \phi) & := & \neg \beta(\phi); \\
\beta(\exists X\ \phi) & := & \exists Z_X, W_X\ (\beta(\phi) \ \wedge\ W_X \neq \emptyset \ \wedge \forall w\ (w \in W_X \rightarrow \\
& & (\forall w'(w' \notin W_X \vee w' \geq w) \rightarrow \forall z (z \notin Z_X \vee z \geq w)))).
\end{array}
$$

It is routine to check that the MCL$[<, \downarrow_0, \downarrow_1, T_0, T]$-sentence ϕ holds in $\langle \mathcal{T}, \downarrow_0, \downarrow_1, <, T_0, T \rangle$ if and only if $\beta(\phi)$ holds in $\langle \mathbb{Z}, < \rangle$.

To complete the proof it suffices to show that MSO$[<]$ over $\langle \mathbb{Z}, < \rangle$ is reducible to MSO$[<]$ over $\langle \mathbb{N}, < \rangle$. To this end, let us denote by $even$ and odd the (definable) unary predicates $\{2n : n \in \mathbb{N}\}$ and $\{2n+1 : n \in \mathbb{N}\}$, respectively. Then, we translate any given formula over $\langle \mathbb{Z}, < \rangle$ into an equi-satisfiable formula over $\langle \mathbb{N}, < \rangle$ by replacing every atomic formula of the form $x < y$ with the formula $(odd(x) \wedge even(y)) \vee (even(x) \wedge even(y) \wedge x < y) \vee (odd(x) \wedge odd(y) \wedge y < x)$. By composing the two translations, we have that MCL$[<, \downarrow_0, \downarrow_1, T_0, T]$ is reducible to MSO$[<]$. ■

The following theorem shows that a similar result holds for the expansion of the TULS with the level 0 predicate T_0 and the equi-column predicate D.

THEOREM 8. — MCL$[<, \downarrow_0, \downarrow_1, T_0, D]$ *over the TULS and* MSO$[<]$ *over* $\langle \mathbb{N}, < \rangle$ *are inter-reducible.*

PROOF. — We first show that MSO$[<]$ over $\langle \mathbb{N}, < \rangle$ is reducible to $\mathrm{MCL}[<, \downarrow_0, \downarrow_1, T_0, D]$ over the expanded TULS. In a way similar to the proof of Theorem 7, we replace MSO$[<]$ with the equivalent logic $\mathrm{MSO}_0[\subseteq, Succ]$ where only second-order variables occur and atomic subformulas are of the forms $X \subseteq Y$ and $Succ(X, Y)$. Then, we denote by D_0 the leftmost upward branch of the 2-refinable TULS. Such a predicate can be defined by a suitable formula in the chain fragment of MSO logic over the TULS. Finally, we translate any MSO$_0[\subseteq, Succ]$-formula ϕ into an equi-satisfiable MCL$[<, \downarrow_0, \downarrow_1, T_0, D]$-formula $\alpha(\phi)$, by following the same construction of Theorem 7. Since D_0 is definable in MCL$[<, \downarrow_0, \downarrow_1, T_0, D]$, we have that MSO$_0[\subseteq, Succ]$ (and hence MSO$[<]$) is reducible to MCL$[<, \downarrow_0, \downarrow_1, T_0, D]$.

As for the opposite reduction, in order to make it possible to check whether two elements of the TULS lie on the same column, we need to encode non-empty chains by suitable elements and sets of elements over $\langle \mathbb{N}, < \rangle$. Since the ordering $<$ of the TULS can be easily defined by a formula in the chain fragment of MSO logic, we can restrict ourselves to the equivalent setup of MCL$[<, \downarrow_0, \downarrow_1, T_0, D]$ where variables are instantiated with non-empty chains over the 2-refinable TULS and atomic formulas are of the forms $X \subseteq Y$ (chain X is included in chain Y), $\downarrow_i (X, Y)$ (X and Y are the singletons $\{x\}$ and $\{y\}$, respectively, and $y = \downarrow_i (x)$), $T_0(X)$ (X is the singleton $\{x\}$, with $x \in T_0$), and $D(X, Y)$ (X and Y are the singletons $\{x\}$ and $\{y\}$, respectively, and x and y belong to the same column). As in the proof of Theorem 7, we existentially close any MCL$[<, \downarrow_0, \downarrow_1, T_0, D]$-formula to be interpreted over $\langle \mathcal{T}, (\downarrow_i)_{i=0}^{k-1}, <, T_0, D \rangle$ and we translate the resulting sentence ϕ into an MSO$[<, neg]$-sentence $\beta(\phi)$ over the structure $\langle \mathbb{Z} \cup \{\infty\}, <, neg \rangle$, where ∞ denotes a special element not belonging to $\mathbb{Z}$ and neg denotes the binary relation $\{(z, -z) : z \in \mathbb{Z}\}$. We shall later show that MSO$[<, neg]$ over $\langle \mathbb{Z} \cup \{\infty\}, <, neg \rangle$ is in its turn reducible to MSO$[<]$ over $\langle \mathbb{N}, < \rangle$.

In order to map sentences of MCL$[<, \downarrow_0, \downarrow_1, T_0, D]$ into sentences of MSO$[<, neg]$, we encode any non-empty chain C by an integer s_C and three subsets Z_C, W_C, and Q_C of $\mathbb{N}$. We denote by P_C the rightmost cover of C, namely, the superset of C that, for each $b \in \mathbb{Z}$, contains exactly one element a_b of the TULS and such that, whenever b is the least integer for which there is a satisfying $a_b \in C$, every descendant of a_b along P_C is of the form c_d, with $c = 2^{b-d}(a+1) - 1$ and $d \leq b$. We must distinguish between two cases: either P_C coincides with the leftmost branch of the TULS (this happens when C is a downward infinite chain lying entirely on the leftmost branch) or there is a minimum index $i \in \mathbb{Z}$ such that $0_i \in P_C$. In the former case, we set $s_C = \infty$, $Z_C = \emptyset$, $W_C = \{i \geq 0 : 0_{-i} \in C\}$, and $Q_C = \{i > 0 : 0_i \in C\}$. In the latter case, we define s_C as the minimum $i \in \mathbb{Z}$ such that $0_i \in P_C$ and we define $Z_C, W_C, Q_C \subseteq \mathbb{N}$ as follows:

– $b \in Z_C$ iff there is a (unique) odd index $a \in \mathbb{N}$ such that $a_{s_C - b} \in P_C$ (namely, $a_{s_C - b}$ is a $\downarrow_1$-successor in the path P_C);

– $b \in W_C$ iff there is a (unique) index $a \in \mathbb{N}$ such that $a_{s_C - b} \in C$ (namely, C intersects the layer $T_{s_C - b}$);

– $b \in Q_C$ iff $b > 0$ and $0_{s_C + b} \in C$ (namely, C intersects the layer $T_{s_C + b}$).

In both cases, the encoding (s_C, Z_C, W_C, Q_C) uniquely determines the non-empty chain C. Switching to logic, we introduce, for each chain variable X, a first-order variable s_X and three second-order variables Z_X, W_X, and Q_X. The translation of any MCL$[<, \downarrow_0, \downarrow_1, T_0, D]$-sentence ϕ into the corresponding MSO$[<, neg]$-sentence $\beta(\phi)$ is inductively as follows:

$$
\begin{array}{rcl}
\beta(X \subseteq Y) & := & W_X \subseteq W_Y \ \wedge \ Q_X \subseteq Q_Y \ \wedge (s_X = s_Y = \infty \vee \\
& & (s_X = s_Y \neq \infty \wedge Z_X = Z_Y) \ \vee (s_X = s_Y \neq \infty \wedge \\
& & \ \exists w \ (w \in W_X \wedge \forall w'(w' \in W_X \rightarrow w \leq w') \wedge \\
& & \qquad \forall z(z \geq w \rightarrow (z \in Z_X \leftrightarrow z \in Z_Y))))); \\
\beta(\downarrow_0 (X, Y)) & := & (s_Y = s_X - 1 \ \wedge \ Z_X = Z_Y \ \wedge \\
& & \ W_X = W_Y = \{0\} \ \wedge \ Q_X = Q_Y = \emptyset) \vee \\
& & \exists w \ (s_X = s_Y \neq \infty \ \wedge \ Z_X = Z_Y \cup \{w-1\} \ \wedge \\
& & \qquad W_X = \{w\} \ \wedge \ W_Y = \{w-1\} \ \wedge \ Q_X = Q_Y = \emptyset); \\
\beta(\downarrow_1 (X, Y)) & := & s_X = s_Y \neq \infty \ \wedge \ Z_X = Z_Y \ \wedge \\
& & \exists w \ (W_X = \{w\} \ \wedge \ W_Y = \{w-1\}) \ \wedge Q_X = Q_Y = \emptyset; \\
\beta(T_0(X)) & := & W_X = \{neg(s_X)\} \ \wedge \ Q_X = \emptyset; \\
\beta(D(X, Y)) & := & s_X \neq \infty \ \wedge \ s_Y \neq \infty \ \wedge \ Z_X = Z_Y \ \wedge \\
& & \exists w \ (W_X = W_Y = \{w\}) \ \wedge \ Q_X = Q_Y = \emptyset; \\
\beta(\phi \wedge \psi) & := & \beta(\phi) \ \wedge \ \beta(\psi); \\
\beta(\neg\phi) & := & \neg\beta(\phi); \\
\beta(\exists X \ \phi) & := & \exists s_X, Z_X, W_X, Q_X(\beta(\phi) \ \wedge \ Z_X \cup W_X \cup Q_X \subseteq \mathbb{N} \ \wedge \\
& & W_X \cup Q_X \neq \emptyset \ \wedge ((s_X = \infty \wedge Z_C = \emptyset) \vee \\
& & \ (s_X \neq \infty \wedge \forall w((w \in W_x \wedge \forall w'(w' \in W_X \rightarrow w \geq w')) \rightarrow \\
& & \qquad \qquad \forall z(z > w \rightarrow z \in Z_X)))).
\end{array}
$$

It is routine to check that for every sentence ϕ, ϕ holds in the TULS expanded with the predicates T_0 and D if and only if $\beta(\phi)$ holds in the structure $\langle \mathbb{Z} \cup \{\infty\}, <, neg \rangle$.

To complete the proof, it remains to show that MSO$[<, neg]$ over $\langle \mathbb{Z} \cup \{\infty\}, <, neg \rangle$ is reducible to MSO$[<]$ over $\langle \mathbb{N}, < \rangle$. First of all, we observe that the structure $\langle \mathbb{Z} \cup \{\infty\}, <, neg \rangle$ can be embedded in $\langle \mathbb{N}, < \rangle$. Then, let us denote by $even$ and odd the (definable) unary predicates $\{2n : n \in \mathbb{N}\}$ and $\{2n+1 : n \in \mathbb{N}\}$, respectively. We can translate any MSO$[<, neg]$-formula ψ into an MSO$[<]$-formula $\rho(\psi)$ by replacing every atomic formula of the form $x < y$ with the formula $(x = 1 \wedge y = 2) \vee (odd(x) \wedge odd(y) \wedge x < y) \vee (x \neq 0 \wedge even(x) \wedge even(y) \wedge y < x)$ and every atomic formula of the form $neg(x, y)$ with the formula $(x = y = 1) \vee (odd(x) \wedge x = y + 1) \vee (x \neq 0 \wedge even(x) \wedge y = x + 1)$. By composing the two translations, we can conclude that MCL$[<, \downarrow_0, \downarrow_1, T_0, D]$ is reducible to MSO$[<]$. ■

In [FRA 06] Franceschet et al. show that the satisfiability problems for the MSO logic over the DULS and the UULS expanded with either the equi-level or the equi-column predicates are not decidable. These undecidability results are proved by reducing several undecidable problems, e.g., the tiling problem over the two-dimensional infinite grid, to satisfiability problems for the corresponding structures. On the positive side, they prove the decidability of the satisfiability problem for the chain fragment (and thus for the path and FO fragments as well) of MSO logic interpreted over the

DULS and the UULS expanded with the equi-level predicate and over the UULS expanded with the equi-column predicate, but they leave open the problem for the DULS expanded with the equi-column predicate. Since the MSO-definability of the DULS and the UULS in terms of the TULS, equipped with the predicate T_0, holds even if we restrict ourselves to interpretations with chain quantifiers only, Theorems 7 and 8 allow us to positively solve such a decision problem[3].

COROLLARY 9. — *The satisfiability problem for* $\mathrm{MCL}[<, \downarrow_0, \downarrow_1, D]$ *(and thus for* $\mathrm{MPL}[<, \downarrow_0, \downarrow_1, D]$ *and* $\mathrm{FO}[<, \downarrow_0, \downarrow_1, D]$ *as well) over the DULS is decidable.*

6. Conclusions

In this paper we explored the relationships between the MSO theory of one successor (the so-called sequential calculus) and various theories of ω-layered structures for time granularity. By taking advantage of logical interpretation as well as of some relaxed variants of it, we reduced the decision/satisfiability problem for MSO logic over $\langle \mathbb{N}, < \rangle$ to that for FO/chain logics over ω-layered structures, and vice versa. On the one hand, these reductions establish interesting links between the sequential calculus and proper fragments of MSO theories of ω-layered structures; on the other hand, they provide an effective solution to the decision/satisfiability problem for expressive theories of time granularity. We intend to compare such a translation-based solution with alternative direct ones in future research work.

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3. As a matter of fact, they allow us to solve the decision problem for the chain/path/FO fragments of MSO logic over the DULS expanded with the vertical successor $\oplus$ as well, since $\oplus$ is FO definable in terms of D [FRA 06].

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